Chapter 21 Some Methods of Integration

21.1 Integration by Substitution(代入求積法)

Changes the variable of integration into another in order to transform the integral into a standard form.

Example 1: Evaluate $\int \frac{1}{(2x+1)^2}\,dx$.

Solution:

Example 2: Evaluate $\int x^3 \sqrt{x^4+5}\,dx$.

Solution:

Example 3: Find $I = \int x\sqrt{1-3x}\,dx$.

Solution:

Example 4: Evaluate $\int \frac{4x+3}{\sqrt{2x^2+3x+5}}\,dx$.

Solution:

Example 5: Evaluate $\int \frac{x}{(2x-1)^{2/3}}\,dx$.

Solution:

Example 6: (a) Expand $(2x-1)^3$ *by the binomial theorem and hence find* $\int (2x-1)^3\,dx$.

(b) Use the substitution $u = 2x - 1$ *to find* $\int (2x-1)^3\,dx$.

Solution:

Class work : Ex 21A # 1- 23(odd), 24-28

21\. 2 Trigonometric Integrations

Example 7: Evaluate (a) $\int \sin 3x\,dx$ *(b)* $\int \cos \frac{1}{2}x\,dx$ *(c)* $\int \sec^2 5x\,dx$

(d) $\int \tan 3x \sec 3x\,dx$ *(e)* $\int \frac{\sin 4x}{\sqrt{2+\cos 4x}}\,dx$

Solution:

Remark: $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$, $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \ldots$

(A) Integration of products of sines and cosines

The following identities are often used to evaluate trigonometric integrals.

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\qquad$ $\sin x \cos y = \frac{1}{2}[\sin(x - y) + \sin(x + y)]$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ $\qquad$ $\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$

$\sin x \cos x = \frac{1}{2}\sin 2x$ $\qquad$ $\cos x \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$

Consider $\int \sin mx \sin nx\, dx$, $\int \sin mx \cos nx\, dx$ or $\int \cos mx \cos nx\, dx$

Example 8: Evaluate a) $\int \sin 2x \sin 8x\, dx$ $\qquad$ b) $\int \sin 3x \cos 5x\, dx$

c) $\int \sin^4 x\, dx$ $\qquad$ d) $\int \cos mx \cos nx\, dx$

Solution:

Consider $\int \sin^m x \cos^n x\, dx$

Case 1: cosine index is odd ($u = \sin x$)

Example 9: Find (a) $I = \int \sin^2 x \cos x\, dx$ and (b) $I = \int \cos^5 x\, dx$.

Solution:

Case 2: sine index is odd (u = cos x)

Example 10: Find (a) $I = \int \sin^3 x \cos^2 x \, dx$ *and (b)* $I = \int \sin^7 \theta \cos^4 \theta \, d\theta$.

Solution:

Case 3: Both indices are even (u = $\cos^2 x$, $\sin^2 x$ or sin x cos x)

Example 11: Evaluate (a) $\int \sin^2 x \cos^2 x \, dx$ *and (b)* $\int \sin^2 x \cos^4 x \, dx$.

Solution:

Case 4: Both indices are odd (u = sin x or cos x)

Example 12: Evaluate $\int \sin^3 x \cos^5 x \, dx$.

Solution:

(B) Integration of products of Tangents and Secants

Consider $\int \tan^m x \sec^n x \, dx$

Case 1: m is odd (u = sec x)

Example 13: Evaluate (a) $\int \tan^3 x \sec^5 x \, dx$ *and (b)* $\int \tan^7 x \sec^4 x \, dx$.

Solution:

Case 2: n is even (u = tan x)

Example 14: Find (a) $I = \int \tan^3 x \sec^2 x\, dx$ *and (b)* $I = \int \tan^4 x \sec^6 x\, dx$

Solution:

(C) Integration of products of Cotangents and Cosecants

Consider $\int \cot^m x \csc^n x\, dx$

Example 15: Find $I = \int \cot^3 x \csc^5 x\, dx.$

Solution:

Remarks: If m is even and n is odd, the integral is very difficult to find and the method is beyond the scope of this book.

Class work: Ex 21B# 1 – 50
Assignment 1: Ex21A#2, 4, 8,14,18; Ex21B#4, 8, 14, 32, 44

21.3 Trigonometric Substitution

Integral involving	use	to obtain
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$\sqrt{a^2 - a^2 \sin^2 \theta} = a \cos \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$\sqrt{a^2 + a^2 \tan^2 \theta} = a \sec \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sqrt{a^2 \sec^2 \theta - a^2} = a \tan \theta$

The range of θ is the principal values of the related trigonometric function.

Example 16: Find $I = \int \frac{dx}{(x^2 + 2)^{3/2}}$.

Solution:

Example 17: Find $I = \int \frac{dx}{\sqrt{9-4x^2}}$.

Solution:

Example 18: Find $I = \int \frac{\sqrt{u^2-16}}{u}du$ *using the substitution* $u = 4\sec\theta$.

Solution:

Class work: Ex 21 C # 1- 16

21.4 Reduction Formulae(歸約公式)

We have established a reduction formula which expresses I_n in terms of I_m. Successive use of such a formula will often allow a given integral to be expressed in terms of a much simpler form.

Class work: Ex 21D #1-4, 6; Rev Ex 21#37 - 50

Assignment 2: Rev Ex 21 #34,36,49