

Chapter 20 Indefinite Integral and its Applications

20.1 Indefinite Integrals

Definition: Integration is the reverse process of differentiation.

Consider $\frac{dy}{dx} = f(x)$ (*)

Denote $\int f(x) dx$ to represent the solution of (*)

i.e. $y = \int f(x) dx$ (indefinite integral of $f(x)$ w.r.t. x)

Example: $\int 4x^3 dx = x^4 + C$

\int → integral sign
 $4x^3$ → integrand (被積函數)
 dx → differential
 $=$ → arbitrary constant (constant of integration) 積分常數
 $x^4 + C$ → primitive function (antiderivative) 原函數

Theorem 1: If f and g are function of x , and k is a constant, then

$$(a) \int kf(x) dx = k \int f(x) dx \text{ where } k \text{ is a non-zero constant.}$$

$$(b) \int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

Proof:

Theorem 2: If $f(x)$ and $g(x)$ are differentiable functions such that $f'(x) = g'(x)$, then
 $f(x) = g(x) + c$ where c is arbitrary constant.

Proof:

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20.2 Integration Formulae for Standard Forms

1. $\int dx = x + C$

2. $\int x^n dx =$

3. $\int \cos \theta d\theta =$

4. $\int \sin \theta d\theta =$

5. $\int \sec^2 \theta d\theta =$

6. $\int \csc^2 \theta d\theta =$

7. $\int \tan \theta \sec \theta d\theta =$

8. $\int \cot \theta \csc \theta d\theta =$

Remark: Formula (2) cannot be applied when $n = -1$.

Discussion of the integral $\int x^{-1} dx$, i.e. $\int \frac{1}{x} dx$, is beyond the scope of this book.

Example 1: Evaluate the following indefinite integrals:

a) $\int 6x dx$

b) $\int (3x^5 + 6x^2 + 8x - 5) dx$

c) $\int (1 + x^2)\sqrt{x} dx$

d) $\int \frac{1}{4} dx$

e) $\int \frac{dx}{x^3}$

f) $\int \frac{8x - 2}{\sqrt[3]{x}} dx$

g) $\int 3\sin x dx$

h) $\int \sin^2 \frac{x}{2} dx$

i) $\int \tan^2 x dx$

j) $\int \frac{\sin x}{\cos^2 x} dx$

Solutions:

Example 2: Let $f(x) = x \cos x$.

- a) Find $f'(x)$. b) Hence, evaluate $\int x \sin x \, dx$.

Solutions:

Class work: Ex 20A # 7 – 34, 36

20.3 Geometrical and Physical Applications

A) Geometrical Applications

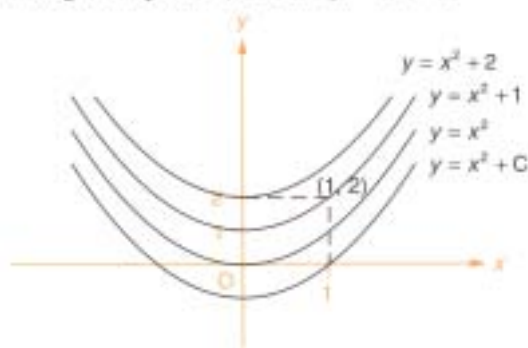
The derivative $\frac{dy}{dx}$ of a curve at a point x is the gradient(slope) of the curve at that point. The geometric interpretation of indefinite integration is to find a curve $y = F(x) + C$ whose gradient at x is given.

Example 3: The point $(1, 2)$ is on a curve, and at any point (x, y) on the curve,

$$\frac{dy}{dx} = 2x. \text{ Find the equation of the curve.}$$

Solutions:

We can now observe the geometric significance of the arbitrary constant C . When C runs through all real constants, $y = x^2 + C$ is a family of parabolas. Each of them can be obtained by shifting the graph of $y = x^2$ through a vertical displacement C (see Fig. 17.1). The particular curve which passes through the point $(1, 2)$ is $y = x^2 + 1$.



Example 4: Find the equation of the curve $y = f(x)$ in each of the following cases:

a) $\frac{dy}{dx} = (x + 1)(3x - 1)$; $y = 2$ when $x = -1$.

b) $\frac{d^2y}{dx^2} = \frac{1}{2\sqrt{x}}$; $y = 10$ and $\frac{dy}{dx} = 0$ when $x = 9$.

Solutions:

B) Physical Applications:

$$s = \int v \, dt, \quad v = \int a \, dt$$

Example 5: A particle is moving along a straight line. At time t , its displacement, velocity and acceleration are s , v and a respectively. Express s in terms of t :

a) $v = 3t + 2$; $s = 0$ when $t = 0$.

b) $a = 2t - 6t^2$; $v = 3$ and $s = 5$ when $t = 1$.

c) $a = -4 \sin t$; $v = 4$ and $s = 7$ when $t = 0$.

Solutions:

Example 6: The curve for which $\frac{dy}{dx} = k(x-p)(x-q)$, where k , p and q are non-zero

constants and $p < q$, has turning points at $(2, 0)$ and $(1, 1)$.

- a) State the values of p and q .*
- b) Determine the value of k .*
- c) Find the equation of the curve.*
- d) Sketch the curve for $-1 \leq x \leq 4$.*

Solutions:

Class work: Ex 20B # 1 – 18, Rev. Ex 20 # 16, 17, 18, 20

Assignment: Rev. Ex 20 # 8, 11, 12, 14, 19