# Chapter 15 Limits and Derivatives

## 15.1 The Idea of Limits

Consider these functions, $f(x) = \dfrac{x^2 - 9}{x - 3}$, $g(x) = x + 3$, $h(x) = \sqrt{9 - x^2}$.

1. Evaluate the given functions for given values of $x$.

| $x$ | 2.9 | 2.999 | 2.999 99 | 3 | 3.000 01 | 3.001 | 3.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | | |
| $g(x)$ | | | | | | | |
| $h(x)$ | | | | | | | |

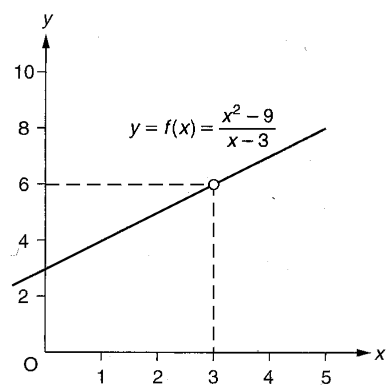
2. What is the range of values of $x$ for which each of the above functions are defined?
   *f(x):*
   *g(x):*
   *h(x):*

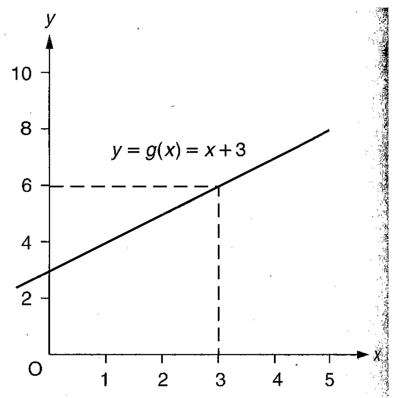
3. What are the values of each of the above functions when
   (a) $x$ approaches 3 from the left, i.e. through values that are less than 3?
   (b) $x$ approaches 3 from the right, i.e. through values that are greater than 3?

We see that $f(x) = \dfrac{x^2 - 9}{x - 3}$ is defined for all values of $x$ except $x = 3$. But as $x$ gets closer and closer to 3, $f(x)$ gets closer and closer to 6. We say that the ***limit of the function f(x)*** when $x$ tends to 3 is 6, and write $\lim_{x \to 3} f(x) = 6$ .

Now compare $f(x) = \dfrac{x^2 - 9}{x - 3}$ and $g(x) = x + 3$. They are identical for all $x$ except $x = 3$.

In addition, $g(x)$ is defined for $x = 3$ and $\lim_{x \to 3} g(x) = 6 = g(3)$. The graph of $y = f(x)$ has a "hole" at $x = 3$; we say that $f(x)$ is discontinuous at $x = 3$. On the other hand, the graph of $y = g(x)$ is a straight line; we say that $g(x)$ is a continuous function.



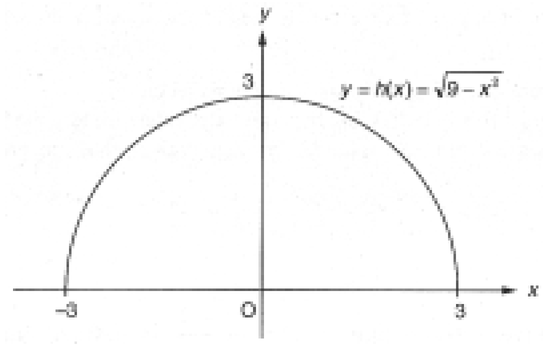

| Definition: If a function $f(x)$ is continuous at $x_0$, then $$\lim_{x \to x_0} f(x) = f(x_0) .$$ |
|---|

The function $h(x) = \sqrt{9 - x^2}$ is undefined for $x < -3$ or $x > 3$. Its graph is a semi-circle. When $x$ approaches 3 from the left, $h(x)$ tends to 0; we say that the *left hand limit* of $h(x)$ exists at $x = 3$ and is equal to 0. We write $\lim_{x \to 3^-} h(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = 0$.

But the *right hand limit* of $h(x)$ does not exist as x is undefined for $x > 3$,

i.e. $\lim_{x \to 3^+} \sqrt{9 - x^2}$ does not exist.

When the left hand limit at a certain point is not equal the right hand limit, the function has no limit at the point. Thus $\lim_{x \to 3} h(x)$ does not exist.




Definition: A function $f(x)$ has limit $\ell$ at $x_0$ if $f(x)$ can be made as close to $\ell$ as we please by taking $x$ sufficiently close to $x_0$. We write

$$\lim_{x \to x_0} f(x) = \ell .$$

Remarks:

1. It requires that when $x$ approaches $x_0$ from both the left-hand side and the right-hand side, $f(x)$ will tend to the same limit $\ell$. In symbolic form,

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \ell.$$

2. The function $f(x)$ is not necessarily defined at $x = x_0$.

*Example 1: Evaluate* $\lim_{x \to 2} (5x + 2)$.

Solutions:

*Example 2: Evaluate* $\lim_{x \to 3} (x^2 + 7x - 5)$.

Solutions:

*Example 3: Evaluate* $\lim_{x \to -2} \frac{x^2 - 1}{x - 1}$.

Solutions:

Classwork: Ex15A #3 - 5

Note: You must ensure that $f(x)$ is continuous at $x = x_0$ before finding limits by direct substitution.

*Example 4: Evaluate* $\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$ .

Solutions:

*Example 5: Evaluate* $\lim_{x \to 1} \frac{x - 1}{\sqrt{x + 3} - 2}$ .

Solutions:

Classwork: Ex15A #6 – 10

*Example 6: Evaluate* $\lim_{x \to 0} \frac{|x|}{x}$.

Solutions:

*Example 7: Evaluate* $\lim_{x \to 0} \frac{1}{x}$.

Solutions:

Remarks: The symbols "$+\infty$" and "$-\infty$" are positive infinity and negative infinity respectively.

Classwork: Ex15A #12, 13, 14

*Example 8: The function f(x) is defined as* $f(x) = \begin{cases} 1 + x & \text{if } x \le -2 \\ -1 & \text{if } -2 < x \le 2 \\ x - 1 & \text{if } x > 2 \end{cases}$.

*(a) Sketch the graph of* $y = f(x)$.

*(b) Find* $\lim_{x \to -2} f(x)$ *and* $\lim_{x \to 2} f(x)$.

*(c) Determine whether f(x) at* $x = -2$ *and at* $x = 2$ *are continuous or not.*

Solutions:

Classwork: Ex 15A # 16, 17

15.2 Theorems on Limits (Assume that $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exist)

If $\lim_{x\to x_0} f(x) = \ell$ and $\lim_{x\to x_0} g(x) = m$, then

(a) $\lim_{x\to x_0} c = c$, where c is a constant

(b) $\lim_{x\to x_0} [c \cdot f(x)] = c \cdot \lim_{x\to x_0} f(x) = c\ell$

(c) $\lim_{x\to x_0} [f(x) \pm g(x)] = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) = \ell \pm m$

(d) $\lim_{x\to x_0} [f(x) \cdot g(x)] = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) = \ell m$

(e) $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{\ell}{m}$, for $m \neq 0$.

15.3 Limits of Trigonometric Functions

| Theorem 1: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, where $\theta$ in radian. |
|---|

Proof:

*Example 9: Evaluate*

(a) $\lim_{\theta \to 0} \dfrac{\theta}{\tan \theta}$ (b) $\lim_{\theta \to 0} \dfrac{\sin 3\theta}{2\theta}$

(c) $\lim_{x \to 0} \dfrac{\sin (x/2)}{\sin (x/3)}$ (d) $\lim_{x \to 0} \dfrac{\sin^2 x}{1 - \cos x}$

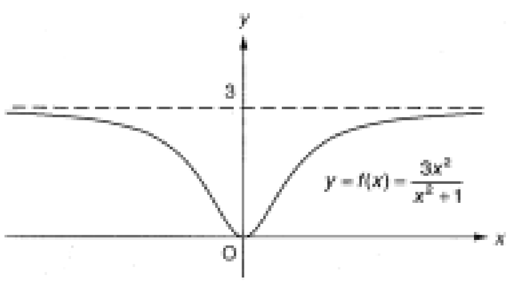
Solutions:

Classwork: Ex 15B #1, 3, 11, 13

15.4 Limits at Infinity

When a variable $x$ increases infinitely through positive values, we write "$x \to \infty$". If $f(x)$ approaches a value L by taking x sufficiently large, we say limit of $f(x)$ at infinity is $L$ and write $\lim_{x \to \infty} f(x) = L$.

Consider $f(x) = \dfrac{3x^2}{x^2 + 1}$, a sketch of the graph of $y = f(x)$ is shown in the following figure.




| $x$ | $\dfrac{3x^2}{x^2+1}$ |
|---|---|
| 10 | 2.970 297 03 |
| 100 | 2.999 700 03 |
| 1 000 | 2.999 997 00 |
| 10 000 | 2.999 999 97 |

We see that as $x$ increases through positive values, the values of $f$(x) get closer and closer to 3. So we say that $\lim_{x \to \infty} \dfrac{3x^2}{x^2 + 1} = 3$.

*Example 10: Evaluate*

(a) $\lim_{x\to\infty} \dfrac{2x^2 - 7}{3x^2 + 2x + 1}$ (b) $\lim_{x\to\infty} \dfrac{8x^2 - 6}{5x + 9}$

(c) $\lim_{x\to\infty} (x - \sqrt{x^2 + 3})$ (d) $\lim_{x\to\infty} \sqrt[3]{\dfrac{16 + x^2}{x(x - 3)}}$

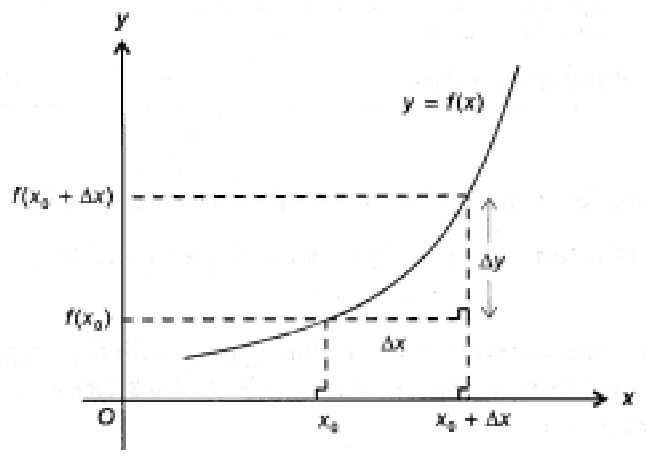
Solutions:

Classwork: Ex 15B #7, 8, 10, 13, 17

15.5 Increments (增量)

The *increment* $\Delta x$ of a variable $x$ is the change in $x$ from $x_0$ to $x_1$ ($\Delta x = x_1 - x_0$).

For a function $y = f(x)$, the increment of $y$ ($\Delta y$) = $f(x_0 + \Delta x) - f(x_0)$.



*Example 11: Find $\Delta y$ for the following functions at given $x_0$ and $\Delta x$:*

(a) $y = 3x + 4$; $x_0 = 2$, $\Delta x = 0.1$

(b) $y = x^2 + 7$; $x_0 = 5$, $\Delta x = -0.01$

(c) $y = \sqrt[3]{x}$; $x_0 = -1$, $\Delta x = 0.0001$

Solutions:

15.6 Derivatives(導數)

Definition: The derivatives of $y = f(x)$ with respect to $x$ is defined as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists.

Notation: $f'(x)$, $\frac{dy}{dx}$, $y'$ or $D_x$

a) The process of finding the derivative of a function is called ***differentiation***(微分法).

b) $f(x)$ is said to be ***differentiable*** at $x = x_0$ if the derivative of $f(x)$ exists at $x = x_0$.

c) The derivative of $f(x)$ at $x = t$ is denoted by $f'(t)$ or $\left.\frac{dy}{dx}\right|_{x=t}$.

d) To obtain the derivative of a function by its definition is called differentiation of function from ***first principle***.

*Example 12: Find the derivative of $y = 3x^2$ w.r.t. $x$ from first principles.*

Solutions:

*Example 13: Differentiate* $y = \dfrac{1}{ax + b}$ *w.r.t. to x from first principles.*

Solutions:

*Example 14: Find the derivatives of* $y = \dfrac{x}{\sqrt{x + 2}}$ *w.r.t. to x.*

Solutions:

Class works: Ex 15C #6, 8

*Example 15: Find the derivatives of* $f(x) = 3x^2 + 1$ *at* $x = 0$*.*

Solutions:

*Example 16: Find the derivatives of f(x) =* $\sqrt[3]{x}$ *at x = 1.*

Solutions:

15.7 Geometrical Interpretation of Derivatives

In the figure, $f(x) = x^2$,

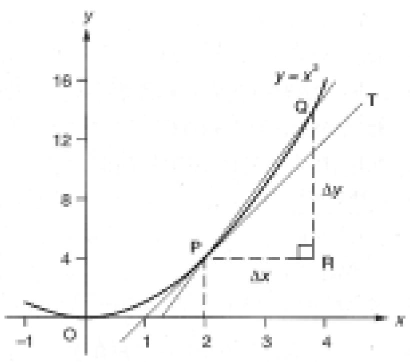



*Fig. 14.11*

$$\frac{dy}{dx} = \text{slope of the tangent at P to the curve } y = f(x)$$

Class works: Rev Ex15 #4 - 8, 10, 12 - 22, 23, 25 - 28, 31, 34

Assignment: Rev Ex 15 #9, 11, 24, 32, 33