A) Factorials

Denote $\boxed{n! = n(n-1)(n-2) \cdots\cdots\cdot 3 \cdot 2 \cdot 1}$ **n : positive integers**

e.g. $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 =$

$5 \times 3! = 5 \times (3 \times 2 \times 1) =$

$$\frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!}$$

Denote $\boxed{0! = 1}$

Classwork

Simplify the following expressions:

(a) $\frac{(n+2)!}{n!} =$

(b) $(n-2)! - (n-3)! =$

(c) $\frac{1}{(n+2)!} - \frac{1}{(n+1)!} - \frac{1}{n!} =$

(d) $\frac{(n+1)! - n! - (n-1)!}{n} =$

B) Simple Permutations

A permutation is an arrangement of items.

e.g. The permutations of the letters A, B, C are ABC, ACB, BAC, , , .

i.e. the number of permutations = $= {}_3P_3$

Note: 1st position 2nd position 3rd position

_____ choices _____ choices _____ choice

$\therefore$ number of permutations = $3 \times \quad \times \quad = 3! =$

Example 1: Two boys and two girls are to form a queue. Find the number of different arrangements.

Solution: The number of different arrangements = P = ! = .

Example 2: In how many ways can the letters of MATHS be arranged?

Solution: Number of ways = P = =

Example 3: How many different numbers of 5 digits may be formed with 1, 3, 5, 7, and 9.

Solution: Number of different numbers can be formed =

C) Combinations

Definition: Given n objects, the number of ways of selecting r of them (r ≤ n) is called the number of combination of n objects taken r at a time. It is denoted by ${}_nC_r$, where

$$\boxed{{}_nC_r = \frac{n!}{r!(n-r)!}, \quad 1 \le r \le n .}$$

Define $\boxed{{}_nC_0 = 1}$

Notes: In combination, the order in which they are taken from the "n" things is not taken into consideration. (It is the main difference between Permutation and Combination.)

Example 4: Given the letters A, B, C, D and E.

Number of letters selected	Outcome	Number of combinations
0	Nothing selected	${}_5C_0$ =
1	A; __; __; __; __.	${}_5C_1 = \frac{5!}{1!(\quad)!}$
2	A B; __ __; __ __; __ __; __ __; __ __; __ __; __ __; __ __; __ __.	${}_5C_2$ =
3	A B C; __ __ __; __ __ __; __ __ __; __ __ __; __ __ __; __ __ __; __ __ __; __ __ __;__ __ __.	${}_5C_3$ =
4	__ __ __ __; __ __ __ __; __ __ __ __; __ __ __ __; __ __ __ __.	${}_5C_4$ =
5	__ __ __ __ __	${}_5C_5$ =

Note : $\mathbf{{}_5C_2 = {}_5C_3,\ {}_5C_1 = {}_5C_4,\ {}_5C_0 = {}_5C_5}$

In general $\boxed{{}_nC_r = {}_nC_{n-r}}$ $\boxed{{}_nC_n = 1}$

Remark :

$$\mathbf{{}_nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot2\cdot3\cdots r}}$$

e.g. ${}_8C_3 = \frac{8!}{3!5!} = \frac{8\times7\times6}{1\times2\times3}$

Example 5: Find the values of

(a) ${}_6C_4 = \frac{6\times5\times4\times3}{1\times2\times3\times4}$

(b) ${}_{13}C_{11}$ =

(c) ${}_6C_2$ =

(d) ${}_{13}C_{10}$ =

(e) ${}_5C_5$ =

(f) ${}_{15}C_3$ =

(h) ${}_{13}C_2$ =

(g) ${}_{21}C_0$ =

Example 6: Simplify

(a) $\frac{{}_{13}C_5}{{}_{13}C_7}$ =

(b) $\frac{{}_{16}C_4\cdot{}_4C_1}{{}_{20}C_5}$ =

Example 7: Find the value of n if ${}_nC_2 = 55$.

Solution: ${}_nC_2 = \frac{n(\quad)}{2} = 55$

$\therefore$

Theorem: $\boxed{{}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}}$

Proof: R. H. S. = ${}_nC_r + {}_nC_{r-1}$

$= \dfrac{n!}{r!(n-r)!} + \underline{\qquad\qquad}$

=

Examples: ${}_5C_3$ = ______ = ;

${}_4C_3$ = ______ = ;

${}_4C_2$ = ______ = .

Thus, we see that ${}_5C_3 = {}_4C_3 + {}_4C_2$.

Example 8: Prove that $n\cdot{}_{n-1}C_r = (r+1)\cdot{}_nC_{r+1}$

Solution: L. H. S. = $n\cdot{}_{n-1}C_r$ R. H. S. = $(r+1)\cdot{}_nC_{r+1}$

= =

Go through Example 1 to 2;pg 79 to 80 on your textbook

D) Binomial Expansions

Note that $(1 + x)^1 = 1 + \quad x$

$(1 + x)^2 = 1 + \quad x + \quad x^2$

$(1 + x)^3 = 1 + \quad x + \quad x^2 + \quad x^3$

$(1 + x)^4 = 1 + \quad x + \quad x^2 + \quad x^3 + \quad x^4$

....

Characteristics for the expressions of $(1 + x)^n$

<1> a total of (   ) terms.

<2> 1st term is _____ and the last term is _____.

<3> a polynomial for x with degrees in _______________ order.

<4> coefficients of (r + 1)th term is _________.

<5> pattern for the coefficients can be generalized by the **Pascal Triangle**.

Power	coefficient
n = 0	1
1	1 1
2	1 2 1
3	1 1
4	1 1
.....	....

In general, we write

Theorem 1:

$$(1 + x)^n = {}_nC_0 + {}_nC_1x + {}_nC_2x^2 + \cdots + {}_nC_rx^r + \cdots + {}_nC_nx^n$$

$$= 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}x^r + \cdots + x^n$$

with $1 \le r \le n$.

Example 9: Expand $(1 + x)^6$.

Solution: $(1 + x)^6 = 1 + (\quad)x + \frac{(\quad)(\quad)}{2}x^2 + \frac{(\quad)(\quad)(\quad)}{1\cdot 2\cdot 3}x^3 + \frac{}{}x^4 + \frac{}{}x^5 + x^6$

$= 1 + \quad x + \quad x^2 + \quad x^3 + \quad x^4 + \quad x^5 + x^6$

Example 10: Expand $(1 + 2y)^6$.

Solution:

$(1 + 2y)^6 = 1 + (\quad)(\quad) + (\quad)(\quad)^2 + (\quad)(\quad)^3 + (\quad)(\quad)^4 +$
$(\quad)(\quad)^5 + (\quad)^6$

=

=

Example 11: Expand $(3x - 1)^5$.

Solution: $(3x - 1)^5 = [-(1 - 3x)]^5 = (-1)^5(\quad)^5 = -(\quad)^5$

$= -[1 + (\quad)(-3x) + (\quad)(\quad)^2 +$
$(\quad)(\quad)^3$

$+(\quad)(\quad)^4 + (\quad)^5]$

=

=

Example 12: If the coefficient of x in the expansion of $(1 + ax)^4$ is 8, find the constant a.

Solution: $(1 + ax)^4 = 1 + (\quad)(\quad) + \cdots\cdots\cdots$

The coefficient of x is = 8

$\therefore$ a =

Example 13: Expand $(0.99)^3$.

Solution: $(0.99)^3 = [1 + (-0.01)]^3$

$= 1 + (\quad)(\quad) + (\quad)(\quad)^2 + (\quad)^3$

=

=

Example 14: (a) Use the first four terms in the expansion of $(1 - 0.01)^4$ to approximate $(0.99)^4$.

(b) Approximte $(0.99)^4$ by using logarithm and calculate the difference between the two methods.

Solution: (a) $(0.99)^4$

$= [1 + (-0.01)]^4$

$= 1 + (\quad)(\quad) + (\quad)(\quad)^2 + (\quad)(\quad)^3$ to first four terms

=

= 0.9606 (correct to 4 sig. fig.)

(b) Put $u = (0.99)^4$

$\log u = \log (0.99)^4$

$= 4 \log (0.99)$

$= 4 (\quad)$

=

$\therefore u = 0.9603$

Thus the difference is 0.9606 – 0.9603 = 0.0003.

Homework: Ex4B 5, 7, 9, 11

Theorem 2: For any positive integer n,

$$(a + b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_ra^{n-r}b^r + \cdots\cdots + {}_nC_nb^n.$$

Proof: $(a + b)^n = [a (1 + \quad)]^n$

$= a^n (1 + \quad)^n$ let a/b = x

$= a^n [1 + {}_nC_1(\quad) + {}_nC_2(\quad)^2 + \cdots + {}_nC_r(\quad)^r + \cdots + {}_nC_n(\quad)^n]$

$= a^n + {}_nC_1 \quad + {}_nC_2 \quad + \cdots + {}_nC_r \quad + \cdots\cdots + {}_nC_n \quad .$

Example 15: Expand $(2 + x^2)^5$.

Solution: $(2 + x^2)^5 = (\quad)^5 + {}_5C_1(\quad)^4(\quad) + {}_5C_2(\quad)^3(\quad)^2 +$

${}_5C_3(\quad)^2(\quad)^3 + {}_5C_4(\quad)(\quad)^4 + (\quad)^5$

=

=

Example 16: Expand $(2t - \frac{1}{t^2})^5$.

Solution: $(2t - \frac{1}{t^2})^5 = (\quad)^5 + 5(\quad)^4(\quad) + 10(\quad)^3(\quad)^2 +$

$10(\quad)^2(\quad)^3 + 5(\quad)(\quad)^4 + (\quad)^5$

=

=

Example 17:

(a) Simplify $(a + b)^5 + (a - b)^5$.

(b) Hence, express $(\sqrt{3} + 1)^5 + (\sqrt{3} - 1)^5$ in the simplest surd form.

Solution: (a) $(a + b)^5 = a^5 + \quad a^4b + \quad a^3b^2 + \quad a^2b^3 + \quad ab^4 + b^5$

$(a - b)^5 = a^5 - \quad a^4b + \quad a^3b^2 - \quad a^2b^3 + \quad ab^4 - b^5$

$\therefore (a + b)^5 + (a - b)^5 = \quad .$

(b) Taking $a = \sqrt{3}$, b = 1; from (a)

$(\sqrt{3} + 1)^5 + (\sqrt{3} - 1)^5 = 2(\quad)^5 + 20(\quad)^3(\quad)^2 + 10(\quad)(\quad)^4$

=

=

Example 18: Find the coefficient of x^8 in the expansion of $(2 - x)^{11}$.

Solution: Coefficient of x^8 = ${}_{11}C$ $(\quad)^{11-3}(\quad)^8$

= 1320

Example 19: Find the 5th term of $(2x - ½)^{11}$.

Solution: The 5th term = ${}_{11}C$ $(\quad)^{11-4}(\quad)^4$

=

Example 20: Find the constant term of $(x^2 - \frac{1}{x})^9$.

Solution: The (r + 1)th term of the expression = ${}_9C$ $(\quad)^{9-r}(\quad)^r$

=

For constant term, = 0

r =

$\therefore$ The constant term = the th term = ${}_9C$

=

Example 21: Find the term which is independent of x in the expansion of $(3x^3 - \frac{1}{x})^{12}$.

Solution:

Example 22:

In the expansion of $(2 + 3x)^n$, the coefficients of x^3 and x^4 are in the ratio 8 : 15.

Solution: Coefficient of x^3 = ( )$^{n-}$ ( )^3

Coefficient of x^4 = ( )$^{n-}$ ( )^4

$\therefore \frac{\qquad}{\qquad} = \frac{8}{15}$

Example 23:

(a) Given n is a positive integer, expand $(1 + tx)^n$ in ascending powers of x as far as the term in x^2.

(b) If the expansion is $1 + 20x + 45t^2x^2 + kx^3 + \cdots\cdots$, calculate n, t and k.

Solution: (a) $(1 + tx)^n = 1 + ntx +$ ( )$^2 + \cdots\cdots\cdots$

= as far as the term in x^2

(b) From (a), $nt = 20$ ····················· (i)

$= 45t^2$ ····················· (ii)

From (i), t =

E) Approximations

Example 24:

(a) Expand $(2 - x)^7$ in ascending powers of x as far as the term in x^4.

(b) Hence, evaluate $(1.98)^7$ correct to 5 decimal places.

Solution: (a) $(2 - x)^7$ = ( )^7 + ( )^6( ) + ( )^5( )^2 + ( )^4( )^3 + ( )^3( )^4 as far as the term in x^4

=

=

$= 128 - 448x + 672x^2 - 560x^3 + 280x^4$

(b) $(1.98)^7 = (2 - 0.02)^7$

From (a), put x = 0.02

$\therefore (1.98)^7 = 128 - 448(\quad) + 672(\quad)^2 - 560(\quad)^3 + 280(\quad)^4$

=

=119.30436

Example 25:

(a) Write down the expansion of $(1 + 2x)(1 - x)^8$ as far as the term in x^4.

(b) Use the expansion to evaluate 1.4×0.8^8 correct to 4 significant figures.

Solution: (a) $(1 + 2x)(1 - x)^8$

$= (1 + 2x)[1 - 8x + \quad x^2 - \quad x^3 + \quad x^4 + \cdots\cdots\cdots]$

$= (1 + 2x)[1 - 8x + \quad x^2 - \quad x^3 + \quad x^4 + \cdots\cdots\cdots]$

$= 1 - 8x + \quad x^2 - \quad x^3 + \quad x^4 + 2x - \quad x^2 + \quad x^3 - \quad x^4 + \cdots\cdots$

= as far as the term in x^4.

(b) $1.4\times 0.8^8 = [1 + 2(\quad)][1 - (\quad)]^8$

From (a), put x =

$\therefore 1.4\times 0.8^8 = 1 -$

=

= 0.2128 (cor. to 4 sig. fig.)

Homework: Ex4C 8, 10, 12

F) Revisional Problem

1.<85-CE-ADD MATHS II-Q.1>

$(ax + \frac{1}{x^2})^n$ is expanded in descending powers of x, where n is a positive integer and a > 0. If

the fourth term of the expansion is independent of x and is equal to $\frac{21}{2}$, find the values of n and x. (5 marks)

2. <86-CE-ADD MATHS II-Q.2>

In the expansion of $(x^2 + 2)^n$ in descending powers of x, where n is a positive integer, the Coefficient of the third term is 40. Find the value of n and the coefficient of x^4.

(5 marks)

3. <87-CE-ADD MATHS II-Q.1>

If the coefficient of x^2 in the expansion of $(1 + x + x^2)^n$ is 21 and n is a positive integer, Find the value of n. (5 marks)

4. <88-CE-ADD MATHS II-Q.1>

Given $(1 + 3x)^4(1 - 2x)^5 = 1 + ax + bx^2$ + higher powers of x, find the values of the Constants a and b. (5 marks)

5. <89-CE-ADD MATHS II-Q.1>

Find the constant term in the expansion of $(1 + x)^{10}(1 - \frac{2}{x})^3$. (5 marks)

6. <90-CE-ADD MATHS II-Q.1>

Given $(1 + 2x - 3x^2)^n = 1 + ax + bx^2$ + terms involving higher powers of x, where n is a positive integer.

(a) Express a and b in terms of n.

(b) If b = 63, find the value of n. (5 marks)

7. <91-CE-ADD MATHS II-Q.1>

Given that $(1 + x + ax^2)^8 = 1 + 8x + k_1x^2 + k_2x^3$ + terms involving higher powers of x.

(a) Express k_1 and k_2 in terms of a.

(b) If $k_1 = 4$, find the value of a.

Hence find the value of k_2. (5 marks)

8. <92-CE-ADD MATHS II-Q.2>

In the expansion of $(1 + 3x)^2(1 + x)^n$, where n is a positive integer, the coefficient of x is 10.

(a) Find the value of n.

(b) Find the coefficient of x^2. (5 marks)

9. <93-CE-ADD MATHS II-Q.3>

Given $(1 + 4x + x^2)^n = 1 + ax + bx^2$ + other terms involving higher powers f x, where n is a positive integer.

(a) Express a and b in terms of n.

(b) If a = 20, find n and b. (6 marks)

10. <94-CE-ADD MATHS II-Q.3>

(a) Expand $(1 - 2x)^3$ and $(1 + \frac{1}{x})^5$.

(b) Find, in the expansion of $(1 - 2x)^3(1 + \frac{1}{x})^5$.

(i) the constant term, and

(ii) the coefficient of x. (5 marks)

11. <95-CE-ADD MATHS II-Q.4>

Given $(x^2 + \frac{1}{x})^5 - (x^2 - \frac{1}{x})^5 = ax^7 + bx + \frac{c}{x^5}$, find the values of a, b and c.

Hence evaluate $(2 + \frac{1}{\sqrt{2}})^5 - (2 - \frac{1}{\sqrt{2}})^5$. (6 marks)

12. <96-CE-ADD MATHS II-Q.2>

It is given that $(1 + ax + bx^2)^6 = 1 + 6x + k_1x^2 + k_2x^3$ + term involving higher powers of x.

(a) Express k_1 and k_2 in terms of a.

(b) If 6, k_1 and k_2 are in A.P., find the value of a. (6 marks)

13. <97-CE-ADD MATHS II-Q.8>

Expand $(1 + x)^n(1 - 2x)^4$ in ascending powers of x up to the term x^2, where n is a positive integer. If the coefficient of x^2 is 54, find the coefficient of x. (7 marks)